Table 1 Integrated errors resulting from linearization of g_1 and g_2 functions

Change in mean anomaly from initial point	Correct value of $\delta\rho$	Error in $\delta\rho$ due to first-order approximation	Error in $\delta\rho$ due to second-order approximation	Error in $\delta\rho$ due to third-order approximation
0	−0.0008626	0	0	0
$\pi/2$	−0.0699590	−0.0041556	0.0002451	0.0000124
π	0.0112128	0.0042206	0.0016468	0.0001375
$3\pi/2$	0.0704753	0.0390562	0.0017051	0.0002118
2π	−0.0008626	...	−0.0076109	0.0002605

figures but actually would require more. The original solution to this problem proposed by Encke involves a series expansion of the perturbed position vector which yields accurate results when the terms are small. A modification[1] of this technique is suggested by the following development.

From inspection of Fig. 1, it is observed that segment a may be expressed as $\mathbf{S}_s \cdot \mathbf{r}_s / r_s$. Further inspection shows that

$$\sin\chi = (1/r)[(\mathbf{S}_s \cdot \mathbf{e}_h)^2 + S_y^2]^{1/2}$$

where $\mathbf{e}_h$ is a unit vector normal to $\mathbf{r}_s$ in the plane of motion of $\mathbf{r}_s$ and in the direction of $\dot{\mathbf{r}}_s$. It is also apparent that the chord c between $\mathbf{r}$ and $[\mathbf{r}_s + \delta h(\mathbf{r}_s/r_s)]$ is given approximately by χr. The angle subtended by this chord is $\frac{1}{2}\chi$. Therefore, the segment b is given by $(\chi r)\sin(\frac{1}{2}\chi)$. For $\chi \leq 10°$, good approximation for segment b is obtained by letting $\sin\chi = \chi$, resulting in

$$\delta h = (1/2r)[(\mathbf{S}_s \cdot \mathbf{e}_h)^2 + S_y^2] + \mathbf{S}_s \cdot (\mathbf{r}_s/r_s) \tag{10a}$$

or

$$\delta\rho = (1/2rr_s)[(\mathbf{S}_s \cdot \mathbf{e}_h)^2 + S_y^2] + (\mathbf{S}_s/r_s) \cdot (\mathbf{r}_s/r_s) \tag{10b}$$

The possibility of linearizing the expression for g_1 and g_2 when $\delta\rho$ is on the order of 0.1 is considered next. If the appropriate expansions and substitutions are performed, the differential equation for $\mathbf{S}_s$ becomes

$$\ddot{\mathbf{S}}_s \sim -\frac{\mu}{r_s^2}\left[\frac{\mathbf{S}_s}{r_s} - 3\delta\rho\left(\frac{\mathbf{r}_s}{r_s} + \frac{\mathbf{S}_s}{r_s}\right) + 6(\delta\rho)^2\frac{\mathbf{r}_s}{r_s} + 0(\delta\rho^3)\right] + \frac{\mathbf{F}}{m} \tag{11}$$

Table 1 illustrates the integrated error introduced by various orders of approximation for a perturbation from a circular reference orbit such that $\delta\rho \leq 0.071$. With reference to this table, the following conclusions regarding linearization are immediate: 1) first-order linearization will not provide sufficient accuracy for this problem; and 2) if second- or third-order terms are retained, the mechanization is only slightly less complex than using the exact formulation called for in Eqs. (6) and (7).

Figure 2 demonstrates the propagation of error in $\delta\rho$ during one orbital period due to the approximation used in developing Eqs. (10). The data presented in Table 1 and Fig. 2 were generated by digital equipment. Figure 3 illustrates the accuracy achievable with analog equipment by using this method. The trajectory shown covers one-half an orbital period, starting at perigee with $\delta\rho = -0.07048$ and finishing at apogee. The data points indicated on the curve were generated from the exact two-body solution for comparison purposes. The agreement is seen to be excellent.

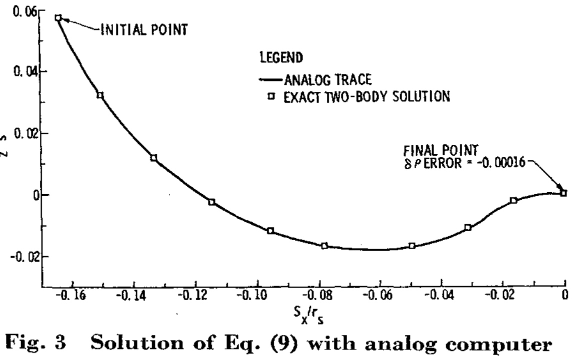

Fig. 3 Solution of Eq. (9) with analog computer

Reference

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Free-Convection Viscoelastic Flow Past a Porous Flat Plate

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THE problem of shear flow of a viscoelastic fluid past a porous flat plate has been considered by Gupta.[1] In the present note an attempt has been made to generalize this problem by taking into account the effect of free-convection when a body force g per unit mass is acting in the negative x direction parallel to the plate. By considering a suitable type of stress-strain relation and assuming the fluid to be semi-incompressible, the solution has been obtained in closed form.

The general stress-strain relation for a viscoelastic fluid is given as

$$\sigma_j{}^i + \tau\tilde{\sigma}_j{}^i = 2\mu(e_j{}^i - \tfrac{1}{3}\Delta\delta_j{}^i) \tag{1}$$

where $\sigma_j{}^i$ is the extra stress tensor (in the sense of Noll[2]), τ is the elastic constant having dimensions of time, μ is the coefficient of viscosity, Δ is dilatation, and $e_j{}^i$ is the rate-of-strain tensor. The term $\tilde{\sigma}_j{}^i$ appearing in Eq. (1) denotes its rate of change, which, following Truesdell,[3] is taken as

$$\tilde{\sigma}_j{}^i = (\partial\sigma_j{}^i/\partial t) + \sigma^i{}_{j,k}v^k + \sigma_j{}^i v_{,k}{}^k - \sigma^{ik}v_{j,k} - \sigma_j{}^k v_{,k}{}^i \tag{2}$$

Similar to the analysis of Gupta, it is assumed that all quantities, except the pressure p, depend on y only. Thus the equations of motion and continuity governing the problem become

$$\rho v(du/dy) = -(\partial p/\partial x) + (d\sigma_y{}^x/dy) - \rho g \tag{3}$$

$$\rho v(dv/dy) = -(\partial p/\partial y) + [d(\lambda\Delta)/dy] + (d\sigma_y{}^y/dy) \tag{4}$$

and

$$d(\rho v)/dy = 0 \tag{5}$$

respectively, where λ denotes the coefficient of bulk viscosity, $\Delta = e_i{}^i = dv/dy$, and $\sigma_x{}^x$, $\sigma_y{}^x$, $\sigma_y{}^y$ are the stress components satisfying Eq. (1).

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Now under the assumption of semi-incompressibility of the fluid, one has

$$\rho - \rho_\infty = -\rho\beta\theta \tag{6}$$

where β is the coefficient of thermal expansion, and ρ_∞ and θ denote density and temperature difference of the fluid far away from the plate, respectively; otherwise ρ is treated to be constant. On considering this fact, Eq. (5) gives

$$v = \text{const} = v_0 \tag{7}$$

and thereby reduces Eqs. (3) and (4) to

$$\rho v_0(du/dy) = -(\partial p/\partial x) + (d\sigma_y{}^x/dy) - \rho g \tag{8}$$

$$0 = -(\partial p/\partial y) + (d\sigma_y{}^y/dy) \tag{9}$$

On using (7), Eq. (1) gives

$$\sigma_x{}^x + \tau[v_0(d\sigma_x{}^x/dy) - 2\sigma_y{}^y(du/dy)] = 0 \tag{10}$$

$$\sigma_y{}^x + \tau[v_0(d\sigma_y{}^x/dy) - \sigma_y{}^y(du/dy)] = \mu(du/dy) \tag{11}$$

$$\sigma_y{}^y + \tau[v_0(d\sigma_y{}^y/dy)] = 0 \tag{12}$$

Equation (12) shows that $\sigma_y{}^y = 0$ is a particular solution of this equation. Hence, putting $\sigma_y{}^y = 0$ in Eq. (11), one gets

$$\sigma_y{}^x + \tau v_0(d\sigma_y{}^x/dy) = \mu(du/dy) \tag{13}$$

From Eq. (8), $\partial p/\partial x$ is constant, since remaining terms in the equation are independent of x, and $\partial^2 p/\partial y\partial x = 0$, by virtue of Eq. (9). Therefore,

$$\partial p/\partial x = \text{const} = -\rho_\infty g \tag{14}$$

Putting this value of $\partial p/\partial x$ in Eq. (8) and using Eq. (6), one has

$$v_0(du/dy) = \beta\theta g + (1/\rho)(d\sigma_y{}^x/dy) \tag{15}$$

Eliminating $\sigma_y{}^x$ between Eqs. (13) and (15), one gets

$$[(\mu/\rho) - \tau v_0{}^2](d^2u/dy^2) - v_0(du/dy) + \beta g\theta + \tau\beta g v_0(d\theta/dy) = 0 \tag{16}$$

The equation of energy in the mechanical units can be written as

$$\rho c v_0(d\theta/dy) = k(d^2\theta/dy^2) + \mu(du/dy)^2 \tag{17}$$

where c and k denote the specific heat and thermal conductivity of the fluid, respectively.

The rigorous solution of the key Eqs. (16) and (17), in general, can be obtained numerically. However, the analysis then would be very complicated, and the physical aspect of the problem would be masked. Therefore, neglecting the viscous dissipation term in Eq. (17), which is justified for slow motion as in the case with free-convection flows, one gets

$$\rho c\, v_0(d\theta/dy) = k(d^2\theta/dy^2) \tag{18}$$

Equations (16) and (18) now are solved under the boundary conditions

$$\begin{matrix} u = 0 \text{ at } y = 0 & u = U_\infty \text{ at } y = \infty \\ \theta = \theta_0 \text{ at } y = 0 & \theta = 0 \text{ at } y = \infty \end{matrix} \tag{19}$$

Before solving these equations, it is interesting to remark that the solution of Eqs. (16) and (18) subject to boundary conditions (19) is physically possible only if $v_0 < 0$ (suction) and $\mu/\rho > \tau v_0{}^2$. That the solution also appears in case of fluid injection ($v_0 > 0$) provided $\mu/\rho < \tau v_0{}^2$, as pointed out by Gupta in his case, does not hold in this case. This is so because the solution of Eq. (18) for $v_0 > 0$ gives positive exponentials.

Taking $v_0 < 0$, it is found that the solution of Eq. (18) subject to (19) is

$$\theta = \theta_0\, e^{-(\rho c v_0/k)y} \tag{20}$$

Eliminating θ between (16) and (20) (taking $v_0 < 0$), one gets

$$(1 - \alpha)(d^2u/d\sigma^2) + (du/d\eta) + A[1 + \sigma\alpha]e^{-\sigma\eta} = 0 \tag{21}$$

where $\eta = \rho v_0 y/\mu$, $\alpha = \tau v_0{}^2\rho/\mu$, $A = \mu g\theta_0\beta/\rho v_0{}^2$, and $\sigma = \mu c/k$ is the Prandtl number.

The solution of Eq. (21) subject to (19) is found to be

$$u = U_\infty[1 - e^{-[\eta/(1-\alpha)]}] - \frac{A(1 + \sigma\alpha)}{[(1 - \alpha)\sigma^2 - \sigma]}[e^{-\sigma\eta} - e^{-[\eta/(1-\alpha)]}] \tag{22}$$

Equation (22) reduces to the familiar asymptotic suction profile of Meredith and Griffith[4] as α and A tend to zero in the limit. It also is seen from the solution that increasing A decreases u, whereas increasing σ decreases the effect of free convection. That this will be so is intuitively clear but only partially, since various factors affect the flow pattern, and hence mathematical corroboration has some interest. Another point that seems to be of physical interest is that the effect of elasticity of the fluid τ will not be perceptible unless v_0 (suction) is present. The reason is that in steady motion the fluid elements do not undergo any change in their state of stress. But as soon as suction comes in, the fluid elements move from one layer of the fluid to the other and thus experience a change in their stress state. The preceding remark, however, will not hold true if in Eq. (11) the third term on the left-hand side, which is an extra term over the material derivative of $\sigma_y{}^x$, also is taken into account. Since in this analysis the author has taken $\sigma_y{}^y = 0$, this term becomes ineffective and the remaining terms correspond to material derivatives only.

The shear stress Tw at the wall is given in the form

$$\frac{Tw}{\rho U_\infty v_0} = \frac{1}{1 - \alpha} + \frac{\sigma A(1 + \sigma\alpha)}{U_\infty[(1 - \alpha)\sigma^2 - \sigma]} \tag{23}$$

which clearly indicates that skin friction increases with the increase of A, as was to be expected.

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Convergence Technique for the Steepest-Descent Method of Trajectory Optimization

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THE steepest-descent approach to trajectory optimization, developed by Kelley[1] and Bryson et al.,[2,3] has proved to be quite successful in overcoming the two-point boundary value problems associated with the calculus of variations. The steepest-descent method is an iterative procedure, requiring repeated forward and backward solutions of sets of differential equations. It is thus of interest to consider techniques for speeding convergence of the iterative process to the optimum trajectory. In this note, a technique re-

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